

# RECOVERING CONDUCTIVITIES ON A LATTICE NETWORK VIA THE MIXED MAP

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ABSTRACT. In this paper we will derive the response matrix for a network in which both boundary currents and boundary voltages are specified. We will briefly explore the properties of this "Mixed Map" matrix as well as discuss a method to recover the conductivities on a lattice electrical network using a specific mixed map. This method can be generalized given any mixed map for a lattice electrical network.

## 1. DERIVATION OF THE MIXED MAP

In a 2007 paper by Jeremiah Jones and Jamie Ramos, [1], it was shown how to derive a response matrix for an electrical network in which both voltage and current are specified on the boundary vertices. In order to keep our paper more self contained, we will briefly repeat the derivation. We will assume knowledge of the work by Curtis and Morrow, [3], particularly the recovery of conductivities throughout.

Let  $\Gamma = (G, \gamma)$  be an electrical network where  $G$  is a connected graph with boundary. Suppose that  $G$  has  $m + n \geq 1$  boundary vertices where  $m, n > 0$  and  $d \geq 1$  interior vertices. Let voltages be specified on the  $m$  boundary vertices, and let currents be specified on the other  $n$  boundary vertices. Order the vertices in the *Kirchhoff* matrix,  $K$ , with the  $m$  vertices first, followed by the  $n$  vertices second, and then the  $d$  interior vertices. Produce the response matrix  $\Lambda = \Lambda_\gamma$  in the usual way (i.e as in [3]).

Now that we have  $\Lambda$ , partition it so that  $\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12}^T & \Lambda_{22} \end{bmatrix}$ . Note that  $\Lambda_{11}$  is  $m \times m$ ,  $\Lambda_{12}$  is  $m \times n$ , and  $\Lambda_{22}$  is  $n \times n$ . Partitioning  $\Lambda$  in this way gives us the following equation:

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12}^T & \Lambda_{22} \end{bmatrix} \begin{bmatrix} v \\ x \end{bmatrix} = \begin{bmatrix} \phi \\ \psi \end{bmatrix} \quad (1)$$

Here  $v$  is the specified voltages on the  $m$  boundary vertices and  $\psi$  is the specified currents on the  $n$  boundary vertices.  $\phi$  is the resulting, previously unknown, currents and  $x$  is the resulting, previously unknown, voltages. Equation (1) implies that

$$\begin{aligned} \Lambda_{11}v + \Lambda_{12}x &= \phi \\ \Lambda_{12}^T v + \Lambda_{22}x &= \psi \end{aligned}$$

Since  $\Lambda_{11}$  and  $\Lambda_{22}$  are principal proper submatrices of  $\Lambda$ , both are invertible, therefore we can uniquely determine  $x$ . Hence,

$$\begin{aligned} x &= -\Lambda_{22}^{-1}\Lambda_{12}^T v + \Lambda_{22}^{-1}\psi \\ \phi &= \Lambda_{11}v + \Lambda_{12}(-\Lambda_{22}^{-1}\Lambda_{12}^T v + \Lambda_{22}^{-1}\psi) \\ &= (\Lambda/\Lambda_{22})v + \Lambda_{12}\Lambda_{22}^{-1}\psi. \end{aligned}$$

So, our desired response matrix, which we will call  $\mathbf{M}$ , referred to as the "Mixed Map", which takes known voltages and currents to previously unknown currents and voltages respectively is

$$M = \begin{bmatrix} \Lambda/\Lambda_{22} & \Lambda_{12}\Lambda_{22}^{-1} \\ -\Lambda_{22}^{-1}\Lambda_{12}^T & \Lambda_{22}^{-1} \end{bmatrix}.$$

$$M \begin{bmatrix} v \\ \psi \end{bmatrix} = \begin{bmatrix} \phi \\ x \end{bmatrix}.$$

## 2. PROPERTIES OF $\mathbf{M}$

**Lemma 2.1.**  $\mathbf{M}$  cannot be a Kirchhoff matrix for an electrical network.

*Proof.* A Kirchhoff matrix is symmetric.  $\mathbf{M}$  is not symmetric and therefore not a Kirchhoff matrix.  $\square$

**Lemma 2.2.**  $\mathbf{M}$  is invertible.

*Proof.* Suppose  $\phi = x = 0$ . Then,

$$u^T K u = \sum_{u_i \in \text{int}G} u_i(Ku)_i + \sum_{u_i \in \partial G} u_i(Ku)_i = 0 + \sum_{\partial G} x_i(Ku)_i + \sum_{\partial G} u_i \phi_i = 0$$

The first sum involving the interior is zero since  $u$  is a harmonic function. The second sum on the right is zero since we specified each potential  $x_i$  to be zero. The third sum is zero since we specified each current  $\phi_i$  to be zero. Therefore,  $u \in \ker \mathbf{K}$ , so  $u$  must be constant. Since  $x = 0$ , it must be that  $u = 0$ . Thus,  $v = 0 = \psi$ . Hence  $\mathbf{M}$  is invertible.  $\square$

In fact, Jones and Ramos [1] explicitly calculated the inverse of  $\mathbf{M}^{-1}$  which is

$$\begin{bmatrix} \Lambda_{11}^{-1} & -\Lambda_{11}^{-1}\Lambda_{12} \\ \Lambda_{12}^T \Lambda_{11}^{-1} & \Lambda/\Lambda_{11} \end{bmatrix}$$

**Lemma 2.3.** The inverse of a positive definite matrix  $\mathbf{A}$  is positive definite.

*Proof.* First note that a positive definite matrix is symmetric and that  $(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1}$ . Now, assume  $\mathbf{A}$  is invertible as well as positive definite. Since  $\mathbf{A}$  is invertible, for all  $x \neq 0$ ,  $\mathbf{A}x = y \neq 0$ , and  $x = \mathbf{A}^{-1}y$ . Since  $\mathbf{A}$  is positive definite, for all  $x \neq 0$ ,  $x^T \mathbf{A}x > 0$ . Therefore,

$$x^T \mathbf{A}x = y^T (\mathbf{A}^{-1})^T y = y^T (\mathbf{A}^T)^{-1} y = y^T \mathbf{A}^{-1} y > 0.$$

$\square$

Notice that since  $\mathbf{M}$  is block skew-symmetric, and not symmetric, we cannot say that  $\mathbf{M}$  is positive semi-definite, but it still holds that  $x^T \mathbf{M}x \geq 0$ .

**Theorem 2.4.**  $x^T \mathbf{M}x \geq 0$  for all  $x \neq \mathbf{0}$

*Proof.* Partitioning the vector  $x$  as needed,

$$\begin{aligned} x^T \mathbf{M} x &= \begin{bmatrix} x_1^T & x_2^T \end{bmatrix} \begin{bmatrix} \Lambda/\Lambda_{22} & \Lambda_{12}\Lambda_{22}^{-1} \\ -\Lambda_{22}^{-1}\Lambda_{12}^T & \Lambda_{22}^{-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= x_1^T (\Lambda/\Lambda_{22}) x_1 - x_2^T \Lambda_{22}^{-1} \Lambda_{12}^T x_1 + x_1^T \Lambda_{12} \Lambda_{22}^{-1} x_2 + x_2^T \Lambda_{22}^{-1} x_2 \end{aligned}$$

The middle two terms are transposes of each other and are single elements so they cancel. We now have

$$x^T \mathbf{M} x = x_1^T (\Lambda/\Lambda_{22}) x_1 + x_2^T \Lambda_{22}^{-1} x_2$$

Since  $\Lambda$  is also a Kirchhoff matrix,  $\Lambda/\Lambda_{22}$  is a response matrix, so  $x_1^T (\Lambda/\Lambda_{22}) x_1 \geq 0$ . Also,  $x_2^T \Lambda_{22}^{-1} x_2 > 0$  since  $\Lambda_{22}^{-1}$  is the inverse of a positive-definite matrix. Therefore,  $x^T \mathbf{M} x \geq 0$ .  $\square$

Note that  $x^T \mathbf{M} x = 0$  if and only if  $x = \begin{bmatrix} \mathbf{c} \\ \mathbf{0} \end{bmatrix}$  where  $\mathbf{c}$  is a constant column vector and  $\mathbf{0}$  is the zero vector.

**Corollary 2.5.** *The diagonal entries of  $\mathbf{M}$  are strictly positive.*

*Proof.* We know the diagonal entries of  $\Lambda/\Lambda_{22}$  are strictly positive since it is a response matrix for  $\Lambda$ . The diagonal entries of  $\Lambda_{22}^{-1}$  are all strictly positive as well. Using the fact that  $\Lambda_{22}$  is positive definite and lemma 2.3,  $e_j^T \Lambda_{22}^{-1} e_j > 0$ , where  $e_j$  is a standard basis vector. This verifies the claim.  $\square$

Below is a brief summary of the properties of the various maps. Here  $\mathbf{H}$  is the Neumann-to-Dirichlet map whose properties were explored in [2].

Comparison of Properties of Response Matrices			
	$\Lambda$	$H$	$M$
Invertible	No	No	Yes
Kernel	Constant Vectors	Constant Vectors	Zero Vector
Symmetric	Yes	Yes	No
$x^T \bullet x \geq 0$ for all $x \neq \mathbf{0}$	Yes	Yes	Yes
Kirchhoff Matrix for a graph?	Yes	No	No
Diagonal Entries	$> 0$	$> 0$	$> 0$
Det. Prin. Prop. Sub-Matrix	$> 0$	$> 0$	?

### 3. RECOVERING CONDUCTIVITIES ON A LATTICE NETWORK

Before beginning the full recovery process we will state a lemma which is true for any circular planar graph.

**Lemma 3.1.** *Let  $G$  be a circular planar graph. If two or more boundary spikes share an interior node, given the response matrix  $\Lambda$ , we can find the conductivities of the boundary spikes.*

*Proof.* We will prove the lemma for two boundary spikes, the generalization to more than two will be obvious. Construct  $\mathbf{M}$  by assigning currents at the two spikes and voltages elsewhere. Call one boundary spike  $b_1$  and the other  $b_2$ . Assign a current of 0 at  $b_1$  and a current of 1 at  $b_2$  while assigning 0 voltage at the rest of the

nodes. Thus, our column vector of assigned currents and voltages is  $\mathbf{x} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ - \\ 1 \\ 0 \end{bmatrix}$ .

Multiplying  $\mathbf{M}$  by  $\mathbf{x}$  we immediately obtain the voltages (namely the bottom two entries of  $\mathbf{M}\mathbf{x}$ ), call them  $v_1$  and  $v_2$ . Since the boundary spike containing  $b_1$  has 0 current, the potential at the adjacent interior node is  $v_1$ . Now for the boundary spike containing  $b_2$  we know the following; the current 1 and the voltages at the the end points  $v_1, v_2$ , so we can find the conductivity  $\gamma$  from Ohm's Law, precisely  $\gamma = \frac{1}{v_2 - v_1}$ . In a similar way, assigning a current of 1 at  $b_1$  and 0 at  $b_2$ , we can find the conductivity of the other boundary spike.  $\square$

We will work with a  $5 \times 5$  square lattice network. The process for larger square and rectangular lattice network will be exactly the same. Now we will show that almost the same method as that used by Curtis and Morrow in [3] for recovering conductivities in a lattice network from  $\Lambda$  can be applied using the mixed map  $M$ . We will use most of the same terminology as in [3]. The main difference in using the mixed map is showing recovery of the boundary conductances, which is what we will fully occupy ourselves with.

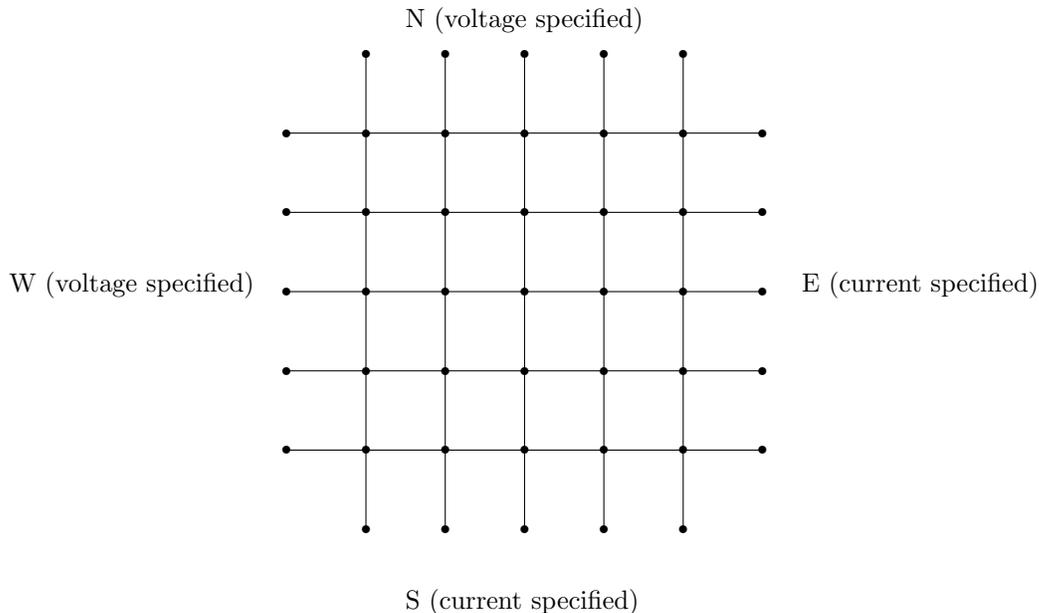


FIGURE 1

Given a lattice network, we will specify voltages on the N and W faces, and specify currents on the S and E faces (See Figure 1). Starting from the right and working left, specify a voltage of one on the boundary vertex adjacent to the corner

boundary vertex and zero voltage everywhere else on the N and W faces. We will apply some unknown currents,  $\mathbf{x}$  and  $\mathbf{y}$ , to the S and E faces respectively, with the hope of the resulting current on the W face all zero and the resulting potential on the S face all zero. If we can achieve this set-up of voltages and currents (See Figure 2), the method of Curtis and Morrow can be applied exactly. The only difficulty occurs in showing that we can always find such voltages  $x$  and  $y$  that will satisfy all of our desired requirements.

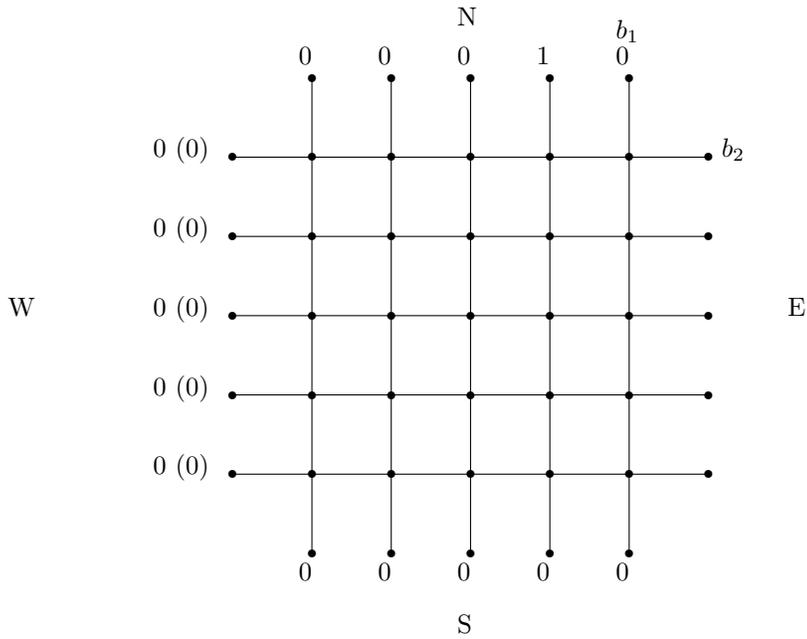


FIGURE 2

Adapting the proof of the previous lemma, we can right away find the conductivities of nodes  $b_1$  and  $b_2$ . In matrix notation our setup is (dividing  $\mathbf{M}$  into smaller blocks):

$$\left[ \begin{array}{cc|cc} M_{11} & M_{12} & & M_{13} & M_{14} \\ M_{21} & M_{22} & & M_{23} & M_{24} \\ \hline M_{31} & M_{32} & & M_{33} & M_{34} \\ M_{41} & M_{42} & & M_{43} & M_{44} \end{array} \right] \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 0 \\ \hline \mathbf{x} \\ \hline \mathbf{y} \end{bmatrix} = \begin{bmatrix} ? \\ \hline \mathbf{0} \\ \hline \mathbf{0} \\ \hline ? \end{bmatrix}$$

We do not care about the ? entries. Multiplying the matrices on the right we have the relevant equations:

$$M_{21} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + M_{23}\mathbf{x} + M_{24}\mathbf{y} = 0$$

$$M_{31} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + M_{33}\mathbf{x} + M_{34}\mathbf{y} = 0$$

In matrix form,

$$\begin{bmatrix} M_{23} & M_{24} \\ M_{33} & M_{34} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} M_2 \\ M_3 \end{bmatrix}$$

where  $M_2 = -M_{21} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$  and  $M_3 = -M_{31} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ .

For a unique solution for  $\mathbf{x}$  and  $\mathbf{y}$  to exist, we must show that  $\begin{bmatrix} M_{23} & M_{24} \\ M_{33} & M_{34} \end{bmatrix}$  is invertible. This argument relies on the geometry of the lattice and Harmonic Continuation. Suppose

$$(1) \quad M_{23}\mathbf{x} + M_{24}\mathbf{y} = 0$$

$$(2) \quad M_{33}\mathbf{x} + M_{34}\mathbf{y} = 0$$

we want to show  $\mathbf{x} = 0$  and  $\mathbf{y} = 0$ . Graphically, equation (1) tell us that there is 0 current on the West side and equation (2) says that there is 0 voltage on the South

side. The absence of  $M_2$  and  $M_3$  implies that there is now 0 voltage on the North side. Also, in our setup, 0 voltages were placed on the West side. By Harmonic Continuation [3] we have  $\mathbf{x} = 0$  and  $\mathbf{y} = 0$  (See figure 3). Therefore, the matrix is invertible, so we have a unique solution for  $\mathbf{x}$  and  $\mathbf{y}$ .

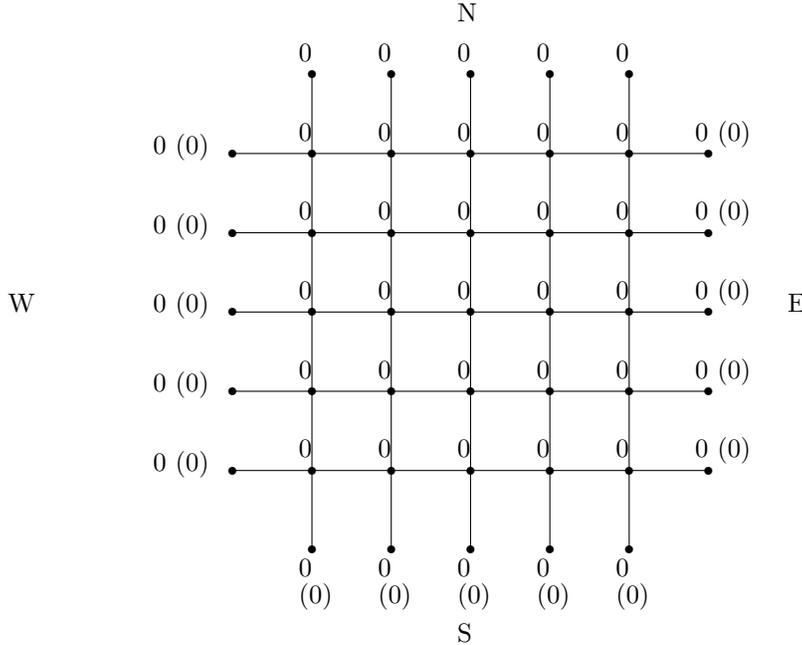


FIGURE 3

Thus, setting currents  $\mathbf{x}$  and  $\mathbf{y}$  on the South and East sides, we now have the same setup of currents and voltages as used by Curtis and Morrow. Hence, using their method (see [3]) we can read off the conductivity of the boundary spike where we placed a voltage of 1. The rest of the recovery is done in the same way as in [3]. To recover conductivities below the main diagonal running from the upper left corner to the bottom right corner, we can apply symmetry (think of the voltages as "currents" and vice versa).

#### 4. GENERALIZED MIXED MAP RECOVERY

The method of recovery using the specific mixed map in the previous section can be easily generalized given any mixed map.

**Theorem 4.1.** *Given a mixed map  $\mathbf{M}$  we can recover conductivities for a rectangular lattice.*

*Proof.* Assume voltages and currents are applied on all faces in a complicated way. Say we have voltage and current specified on every side of the network. The analogue for simpler voltage/current arrangements will be obvious. Again, we will start with the north side working from the right to the left, recovering boundary spikes. Recovering inner conductivities will be exactly the same as [3]. Call the boundary

spike on the north side farthest right  $b_1$ . Assume that voltage is applied at  $b_1$ . We leave it to the reader to verify that if current is applied instead, the recovery process will be nearly identical.

Where appropriate, according to the given mixed map specify the voltages  $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ,

$\mathbf{0}$ ,  $\mathbf{0}$ , and some, to be found, voltage  $\mathbf{x}$  on the N,W,S,E sides respectively moving counterclockwise. Similarly apply some, to be found, currents  $\mathbf{y}$ ,  $\mathbf{0}$ ,  $\mathbf{z}$ ,  $\mathbf{c}$  on the N,W,S,E sides respectively moving counterclockwise. To achieve the setup of Curtis and Morrow we wish to have 0 voltage and current on the West side as well as 0 voltage on the South side. In matrix terms (having the 4 voltages (N,E,S,W) and then the 4 currents in the first column vector and opposite on the right) and partitioning  $\mathbf{M}$  as needed,

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & | & M_{15} & M_{16} & M_{17} & M_{18} \\ M_{21} & M_{22} & M_{23} & M_{24} & | & M_{25} & M_{26} & M_{27} & M_{28} \\ M_{31} & M_{32} & M_{33} & M_{34} & | & M_{35} & M_{36} & M_{37} & M_{38} \\ M_{41} & M_{42} & M_{43} & M_{44} & | & M_{45} & M_{46} & M_{47} & M_{48} \\ \hline M_{51} & M_{52} & M_{53} & M_{54} & | & M_{55} & M_{56} & M_{57} & M_{58} \\ M_{61} & M_{62} & M_{63} & M_{64} & | & M_{65} & M_{66} & M_{67} & M_{68} \\ M_{71} & M_{72} & M_{73} & M_{74} & | & M_{75} & M_{76} & M_{77} & M_{78} \\ M_{81} & M_{82} & M_{83} & M_{84} & | & M_{85} & M_{86} & M_{87} & M_{88} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ \hline \mathbf{0} \\ \hline \mathbf{0} \\ \hline \mathbf{x} \\ \hline \mathbf{y} \\ \hline \mathbf{0} \\ \hline \mathbf{z} \\ \hline \mathbf{c} \end{bmatrix} = \begin{bmatrix} ? \\ \hline \mathbf{0} \\ \hline ? \\ \hline ? \\ \hline \mathbf{0} \\ \hline \mathbf{0} \\ \hline \mathbf{0} \\ \hline ? \end{bmatrix}$$

Proceeding as in the previous section, the relevant system of equations is

$$(3) \quad M_{24}\mathbf{x} + M_{25}\mathbf{y} + M_{27}\mathbf{z} + M_{28}\mathbf{c} = M_2$$

$$(4) \quad M_{54}\mathbf{x} + M_{55}\mathbf{y} + M_{57}\mathbf{z} + M_{58}\mathbf{c} = M_5$$

$$(5) \quad M_{64}\mathbf{x} + M_{65}\mathbf{y} + M_{67}\mathbf{z} + M_{68}\mathbf{c} = M_6$$

$$(6) \quad M_{74}\mathbf{x} + M_{75}\mathbf{y} + M_{77}\mathbf{z} + M_{78}\mathbf{c} = M_7$$



Where  $M_j = -M_{j1}$   $\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ . To uniquely determine  $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{c}$  we must show  $\begin{bmatrix} M_{24} & M_{25} & M_{27} & M_{28} \\ M_{54} & M_{55} & M_{57} & M_{58} \\ M_{64} & M_{65} & M_{67} & M_{68} \\ M_{74} & M_{75} & M_{77} & M_{78} \end{bmatrix}$

is invertible. Setting  $M_2, M_5, M_6, M_7 = 0$  and interpreting the equations (3)-(6) graphically we obtain a similar setup to figure 3. Using Harmonic continuation,  $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{c} = 0$  and so the matrix is invertible. Thus setting voltages and currents  $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{c}$  appropriately, a setup similar to figure 2 is achieved so we can find the conductivity of the boundary spike where we placed a voltage of 1. Thus, we can continue in the way of the previous section and the method presented in [3] to recover the rest of the conductivities.  $\square$

## 5. FURTHER RESEARCH

There is still much that can be done in the way of research for the mixed map. Much less is known about it than the Dirichlet-to-Neumann map  $\mathbf{A}$ , or even the Neumann-to-Dirichlet map  $\mathbf{H}$  as seen in [2]. Some suggestions for further research topics that may be of particular interest are

- Recovery techniques for circular planar graphs other than the lattice.
- A possible analogue of the determinant connection formula.
- Investigation of the eigenvalues and/or eigenvectors of  $\mathbf{M}$ .
- A graphical interpretation of the entries of  $\mathbf{M}$  (e.g. the fact that the upper left block of  $\mathbf{M}$  is a response matrix of some graph)

## REFERENCES

- [1] Jermiah Jones, Jamie Ramos. “The Simplest Mixed Problem” 2007.
- [2] Nate Bottman, James McNutt. “On the Neumann-to-Dirichlet Map” 2007.
- [3] Curtis, B., and James A. Morrow. “Inverse Problems for Electrical Networks.” Series on applied mathematics – Vol. 13. World Scientific, ©2000.